Two kinds of extreme black holes and their classification

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Abstract

According to different topological configurations, we suggest that there are two kinds of extreme black holes in the nature. We find that the Euler characteristic plays an essential role to classify these two kinds of extreme black holes. For the first kind of extreme black holes, Euler characteristic is zero, and for the second kind, Euler characteristic is two or one provided they are four dimensional holes or two dimensional holes respectively.

PACS number(s): 04.70.Dy, 04.20.Gz, 04.62.+v.

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Based upon the topological arguments between the Reissner-Nordström (RN) extreme black hole (EBH) and nonextreme black hole (NEBH), Hawking et. al.[1,2] claimed that the EBH is a different object from its nonextreme counterpart and the Bekenstein-Hawking (BH) formula of the entropy fails to describe the entropy of EBH. A RN EBH has zero entropy, despite the nonzero area of the event horizon. Their result had been extended to two-dimensional black holes [3].

Contrary to the above results, starting from a grand canonical ensemble, Zaslavskii shew that a RN black hole can approach the extreme state as closely as one likes in the topological sector of nonextreme configuration [4,5]. The thermodynamical equilibrium can be fulfilled at every stage of this limiting process and the BH formula of entropy is still valid for the final RN EBH. Zaslavskii also found that the limiting geometry of RN black hole can be described by the Bertotti-Robinson (BR) spacetime.

According to their statements, the RN EBHs of Hawking et. al. and that of Zaslavskii have very different properties. The first kind suggested by Hawking et. al. is the original EBH. It has zero entropy, infinite proper distance l between the horizon and any fixed point, in particular, purely extreme topology. This kind of RN EBH cannot be formed in gravitational collapse, or assimilating infalling charged particle and shell [6] from nonextreme RN black hole. It can only arise through pair creation from the beginning of the universe [1]. The second kind suggested by Zaslavskii is a quite different object which is got by first adopting the boundary condition $r_+ = r_B$, where r_+ is the event horizon and r_B is the boundary of the cavity, and then the extreme condition $Q = M$. It's entropy satisfies BH formula like NEBH. It has finite proper distance l and is still in the topological sector of nonextreme configuration. These results naturally lead one to an impression that there are two kinds of RN EBHs, both satisfying the extreme condition but with different characters in the nature. It is of interest to ask whether these results can be extended to more general, not only include RN black hole, but also include other black holes (at least include two-dimensional (2D) or four-dimensional (4D) black holes). Are there two kinds of EBH in the nature? If there are, how to classify them? This paper envolves from an attempt to answer this problem and give a classification to two kinds of general extreme holes.

Since the close relation between the black hole intrinsic thermodynamics and its topology[7,8] and the topological difference between the first kind and second kind EBHs, it is natural to address this problem from the beginning by their topological characters. We will

prove the topological properties play an essential role in the classification of these two kinds of EBHs.

We study the four-dimensional (4D) black holes first.

I. 4D black holes.

The Euclidean metric form of 4D spherical black holes reads

$$\mathrm{d}s^2 = e^{2U(r)}\mathrm{d}t^2 + e^{-2U(r)}\mathrm{d}r^2 + R^2\mathrm{d}\Omega^2 \tag{1}$$

Using the Gauss-Bonnet (GB) theorem and the boundary condition, one finds the GB action

$$\begin{aligned} S_{GB} &= \frac{1}{32\pi^2}\int_M \varepsilon_{abcd}R^{ab}\wedge R^{cd} \\ &= \frac{1}{4\pi^2}\Big(\int_{\partial V} - \int_{\partial M}\Big)\omega^{01}\wedge R^{23} \end{aligned} \tag{2}$$

and the topological parameter, the Euler characteristic χ takes the form[7,8]

$$\chi = \frac{\beta}{2\pi}[(2U'e^{2U})(1-e^{2U}R'^2)]_{r_+}^{r_0} \tag{3}$$

where $\beta = 4\pi[(e^{2U})'_{r_+}]^{-1}$ is the inverse temperature.

A. RN black hole

The metric is in the form of Eq(1), but

$$e^{2U(r)} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, R^2 = r^2 \tag{4}$$

The extreme case corresponds to $M = Q$. As was pointed out by Ref[1], for the first kind of extreme RN hole, due to infinitely far away of the horizon location, there are no conical singularities, which corresponds to no fixing imaginary time period β, then the Euler characteristic $\chi = 0$[2]. For the second kind of extreme RN black hole, the Euler characteristic has not been calculated before. Directly applying Eq(3) and adopting the boundary condition limit ($r_+ \to r_B$) first and then the extreme condition limit ($M \to Q$) afterwards, we find

$$\begin{aligned} \chi &= [\frac{\beta}{\pi r^3}(M - \frac{Q^2}{r})(2M - \frac{Q^2}{r})]_{r=r_+=r_B}|_{extr} \\ &= \frac{4\pi r_B^6(r_B - M)}{2\pi r_B^6(r_B - M)}|_{extr} = 2 \end{aligned} \tag{5}$$

where we have considered $\beta = \dfrac{4\pi r_+^3}{r_+^2 - Q^2}$, and substract the influence of the asymptotically flat spaces as in ref.[7,8]. The value for the second kind of extreme RN hole is in agreement with that of the nonextreme cases.

B. 4D dilaton black hole.

We extend our discussions to a more general case, namely, the dilaton black hole. The metric is still in the form of Eq(1), but

$$
\begin{aligned}
e^{2U} &= (1-\frac{r_+}{r})(1-\frac{r_-}{r})^{(1-a^2)/(1+a^2)} \\
R^2 &= r^2[1-\frac{r_-}{r}]^{2a^2/(1+a^2)} \\
2M &= r_+ + \frac{1-a^2}{1+a^2}r_- \\
Q^2 &= \frac{r_+ r_-}{1+a^2}
\end{aligned} \tag{6}
$$

this solusion reduces to the RN case when $a=0$ and corresponds to the black hole obtained from string theory [9] when $a=1$. We focus our attention on the case $0<a\leq 1$.

The Euler characteristic χ for the first kind of EBH has been calculated in [7] and got $\chi=0$. Due to nonzero $[(e^U R')^2]_{extr}|_{r=r_+}$, if we first take extreme limit and then approach the horizon, some pecular outcome will emerge. To overcome this difficulty and obtain a unique and satisfactory result of χ, they add an inner boundary $r_0=r_++\epsilon$ and set $\epsilon\to 0$ at the end of calculation.

However, for the second kind of extreme dilaton hole, if taking boundary limit ($r=r_+=r_B$) first and then imposing the extreme condition ($r_+=r_-$) afterwards, we find $[(e^U R')^2|_{r=r_+=r_B}]_{extr}=0$, and

$$
\begin{aligned}
\chi &= [\frac{\beta}{2\pi}2U'e^{2U}]_{r_+=r_B}|_{extr} \\
&= \frac{4\pi r_B}{2\pi r_B}[\frac{r_B(r_B-r_-)}{r_B(r_B-r_-)}]^{(1-a^2)/(1+a^2)}|_{extr} = 2
\end{aligned} \tag{7}
$$

This value is in consistent with that of the nonextreme dilaton case[8].

C. Kerr black hole.

The metric of the Kerr black hole reads

$$
\begin{aligned}
\mathrm{d}s^2 &= -\frac{\triangle}{\Sigma^2}[\mathrm{d}t - a\sin^2\theta \mathrm{d}\phi]^2 + \frac{\sin^2\theta}{\Sigma^2}[(r^2+a^2)\mathrm{d}\phi - a\mathrm{d}t]^2 \\
&\quad + \frac{\Sigma^2}{\triangle}\mathrm{d}r^2 + \Sigma^2\mathrm{d}\theta^2
\end{aligned} \tag{8}
$$

where

$$
\triangle = r^2 - 2Mr + a^2, \Sigma^2 = r^2 + a^2\cos^2\theta, a = J/M \tag{9}
$$

M, J are the mass and angular momentum. The event horizon and the Cauchy horizon locate at $r_+ = M + \sqrt{M^2 - a^2}, r_- = M - \sqrt{M^2 - a^2}$, respectively. The extreme case corresponds to $M = a$.

For the first kind of Kerr EBH, expending the metric coefficients near $r = r_+$, introducing $r - r_+ = r_B \rho^{-1}$, we have

$$\mathrm{d}s^2 = \Sigma_B^2 \rho^{-2} \{ -\frac{r_B^2}{\Sigma_B^4} [\mathrm{d}t - a \sin^2 \theta \mathrm{d}\phi]^2 + \frac{\rho^2 \sin^2 \theta}{\Sigma_B^4} [(r_B^2 + a^2) \mathrm{d}\phi - a \mathrm{d}t]^2 + \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\theta^2 \} \quad (10)$$

The horizon satisfies the condition

$$\triangle = (r_B^2 + a^2) f = r_B^2 \rho^{-2} = 0 \quad (11)$$

where

$$f = \frac{(r - r_+)^2}{r^2 + a^2} \quad (12)$$

The horizon of the first kind of Kerr EBH locates at $\rho = \infty$. The proper distance between $\rho = \infty$ and other $\rho < \infty$ is infinite. The infinite horizon removes the conical singularity and makes the imaginary time β arbitrary. Applying the arguments in [7,2], it leads unambiguously to $\chi = 0$.

We now turn to discuss the second kind Kerr EBH. To get the Euler characteristic, we investigate the metric corresponding to this kind of Kerr EBH first. Putting the nonextreme Kerr black hole in a cavity, the equilibrium condition is

$$\beta = \beta_0 [f(r_B)]^{1/2}, T_0 = 1/\beta_0 = \frac{1}{4\pi} f'(r_+) = \frac{\sqrt{M^2 - a^2}}{2\pi (r_+^2 + a^2)} \quad (13)$$

where $f = \dfrac{\triangle}{r^2 + a^2} = \dfrac{(r - r_+)(r - r_-)}{r^2 + a^2}$ and r_B is the cavity boundary. Choosing

$$r - r_+ = 4\pi T_0 b^{-1} (\sinh^2 x/2), b = \frac{f''(r_+)}{2} \quad (14)$$

and expending $f(r) = 4\pi T_0 (r - r_+) + b(r - r_+)^2$ near r_+, we find that in the extreme limit $r_+ = r_- = r_B (M = a), b = \dfrac{1}{r_B^2 + a^2}$, the metric becomes

$$\begin{aligned} \mathrm{d}s^2 \quad = \quad & -\Sigma_B^2 \sinh^2 x \mathrm{d}t_1^2 + \Sigma_B^2 \mathrm{d}x^2 + \Sigma_B^2 \mathrm{d}\theta^2 \\ & + \frac{\sin^2 \theta}{\Sigma_B^2} [(r_B^2 + a^2) \mathrm{d}\phi - a \sinh x \sqrt{r_B^2 + a^2} \mathrm{d}t_1]^2 \end{aligned} \quad (15)$$

where time is normalized according to $t_1 = 2\pi T_0 t, \mathrm{d}x = b \mathrm{d}l^2$, and $\Sigma_B^2 = r_B^2 + a^2 \cos^2 \theta$. Eq(15) is the generalization of the Bertotti-Robinson (BR) spacetime [10] in the 4D non-spherical case.

The horizon of the extreme Kerr black hole can be detected by

$$\triangle = (r_B^2 + a^2)f = (r_B^2 + a^2)\frac{f'(r_+)}{4}(b^{-1}\sinh^2 x) = 0 \tag{16}$$

which locates at finite x, (say $x = 0$). So the proper distance between the horizon and any other fixed point is finite. By means of the formula of χ[8] and the extreme condition ($r_+ = M = a$), we obtain

$$\begin{aligned} \chi &= \frac{Mr_+(r_+ - M)}{4\pi^2}\int_0^{\beta_0} \mathrm{d}\tau \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \frac{(r_+^2 - 3M^4\cos^4\theta)}{(r_+^2 + M^2\cos^2\theta)^3}\sin\theta \mathrm{d}\theta \\ &= \frac{2}{\pi}\beta_0(r_+ - M)\frac{Mr_+}{(r_+^2 + M^2)^2} = 2 \end{aligned} \tag{17}$$

which is in agreement with that of the nonextreme case [8].

II. 2D black holes.

The formula of the Euler characteristic in 2D cases is [11]

$$\chi = \frac{1}{2\pi}\int R_{1212} e^1 \wedge e^2 \tag{18}$$

A. 2D charged dilaton black hole.

The metric of this black hole is [12,13]

$$\mathrm{d}s^2 = -g(r)\mathrm{d}t^2 + g(r)^{-1}\mathrm{d}r^2 \tag{19}$$

$$g(r) = 1 - 2me^{-\lambda r} + q^2 e^{-2\lambda r} \tag{20}$$

$$e^{-2\phi} = e^{-2\phi_0}e^{\lambda r}, A_0 = \sqrt{2}qe^{-\lambda r} \tag{21}$$

where m and q are the mass and electric charge of the black hole respectively. $m = q$ corresponds to the extreme case. Applying Eq(18) and substract the asymptotically flat space's influence, the Euler characteristic for the NEBH reads

$$\chi = -\frac{\beta}{2\pi}[-\lambda m e^{-\lambda r} + \lambda q^2 e^{-2\lambda r}]_{r_+} = 1 \tag{22}$$

where $1/\beta = g'(r_+)/4\pi$ [12].

We now extend above calculation of χ to two kinds of EBHs. For the first kind of EBH, taking the extreme condition first

$$\chi = -\frac{1}{2\pi}\beta_0[-m\lambda e^{-\lambda r} + m^2\lambda e^{-2\lambda r}]_{r_+} \tag{23}$$

and considering for the original EBH $r_+ = \frac{1}{\lambda}\ln m$, we have

$$\chi = 0 \tag{24}$$

But for the second kind of EBH, adopting the boundary condition first and then the extreme condition, we obtain

$$\begin{aligned}\chi &= -\frac{1}{2\pi}\beta_0[-m\lambda e^{-\lambda r} + q^2\lambda e^{-2\lambda r}]_{r_+=r_B}|_{extr} \\ &= \frac{2\pi(m+\sqrt{m^2-q^2})\lambda\sqrt{m^2-q^2}}{2\pi\lambda\sqrt{m^2-q^2}(m+\sqrt{m^2-q^2})}|_{extr} = 1\end{aligned} \tag{25}$$

We find the same result as that of the NEBH.

B. 2D Lowe-Strominger black hole.

The metric has the same form as Eq(19), but [14]

$$g(r) = \lambda^2 r^2 - m - \frac{J^2}{4r^2} \tag{26}$$

$$A_0 = -\frac{J}{2r^2} \tag{27}$$

$$e^{-2\phi} = r \tag{28}$$

Using Eq(18), the Euler characteristic for this NEBH is

$$\chi = -\frac{\beta_0}{2\pi}[-\lambda^2 r + \frac{J^2}{4r^3}]_{r_+} = 1 \tag{29}$$

For the first kind of EBH, by taking the extreme limit $\lambda J = m$ first,

we find

$$\chi = -\frac{\beta_0}{2\pi\lambda^2}[-\lambda^4 r + \frac{m^2}{4r^3}]_{r_+} \tag{30}$$

and then using $r_+^2 = \frac{m}{2\lambda^2}$ for the original EBH, we get

$$\chi = 0 \tag{31}$$

However, for the second kind of EBH, using the same treatment as before, we find

$$\chi = -\frac{\beta_0}{2\pi}[-\lambda^2 r + \frac{J^2}{4r^3}]_{r_+=r_B}|_{extr} = 1 \tag{32}$$

which agrees to Eq(29) of the NEBH.

According to above calculations, in summary, we suggest that there are two kinds of 4D and 2D EBHs in the nature. The first kind of EBHs is the original EBH. They can be

obtained in mathematics by first taking the extreme limit and then the boundary limit. The entropy of this kind of EBH is zero. The second kind of EBHs still holds the topological configuration of NEBH. They can be obtained in mathematics by doing the other way round, i.e. taking the boundary limit first and the extreme limit afterwards. The entropy of this kind of EBHs satisfies the BH formula. These two kinds of EBHs have different intrinsic thermodynamics. We have shown that the Euler characteristic plays an essential role to classify these two kinds of EBHs. For the first kind, Euler characteristic is zero; and for the second kind, the Euler characteristic equals to two or one provided they are 4D or 2D EBHs respectively.

This work was supported in part by NNSF of China. B.Wang would like to thank helpful discussions with Prof. Randjbar when he visited ICTP.

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